Journal of Computational and Applied Mathematics 235 (2011) 2897–2903

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Computing the moments of order statistics from nonidentically distributed phase-type random variables

Xuan Zhang *, Zhenting Hou

School of Mathematics, Central South University, Changsha, Hunan, PR China

ARTICLE INFO

Article history:
Received 25 March 2010
Received in revised form 25 October 2010

Keywords:
Order statistics
Moments
Phase-type distribution

ABSTRACT

In this paper, we derive a recurrence relation for the single moments of order statistics (o.s.) arising from n independent nonidentically distributed phase-type (PH) random variables (r.v.'s). This recurrence relation will enable one to compute all single moments of all o.s. in a simple recursive manner.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be independent r.v.'s with distribution functions (d.f.'s) $F_1, F_2, \ldots, F_n$ respectively and let $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$ denote the corresponding o.s. obtained by arranging the nX_i's in increasing order of magnitude. The theory of o.s. has attracted much attention during the last three decades because of its wide applications in many fields of statistical inference and probability (see [1–4]). Recently, much work has been done on o.s. from nonidentically distributed variables. The interest is concentrated on the recurrence relations for the moments of o.s. Balakrishnan [5] established several recurrence relations for the moments of o.s. from nonidentically exponential and right-truncated exponential distributions by exploiting a basic relation between the probability density function and the d.f. On the other hand, Bapat and Beg [6] derived recurrence relations for the moments of o.s. by making use of the properties of permanents of matrices. Abdelkader [7] applied the same procedure on o.s. from nonidentical Erlang variables.

In this paper, we consider the case where $X_1, X_2, \ldots, X_n$ are independent and nonidentically distributed r.v.'s having PH distributions. PH distributions have been introduced by Neuts as a generalization of the exponential and Erlang distribution; see [8]. They have been used in a wide range of stochastic modeling applications in areas as diverse as reliability theory, queueing theory, survival analysis, and biostatistics (see [9]).

The paper is organized as follows. Section 2 introduces some preliminaries. In Section 3, we derive the kth moments $\mu_{n:n}^{(k)} = E(X_{n:n}^k)$ and $\mu_{1:n}^{(k)} = E(X_{1:n}^k)$ of the maximum and the minimum of a sample of size n from the PH distribution. A recurrence relation is presented in Section 4 which enables one to compute the kth moments of all o.s. (e.g. $\mu_{r:n}^{(k)}$ for all $r \leqslant n$) in a simple recursive manner by using only the kth moment of the minimum.

2. Preliminaries

In this section, we introduce some preliminaries that will be frequently used in this paper.

* Corresponding author.
E-mail address: zxshuxue@yahoo.com.cn (X. Zhang).

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doi:10.1016/j.cam.2010.11.012

Definition 2.1. A probability distribution $F(\cdot)$ on $[0, \infty)$ is said to be a PH distribution with representation (α, T) if it is the distribution of the time until absorption in a finite-state Markov process on the states $\{1, 2, \ldots, m+1\}$ with generator

$$Q = \begin{pmatrix} T & \eta \\ \mathbf{0} & 0 \end{pmatrix}$$

and initial distribution (α, α_{m+1}), where α is a row vector of dimension m, η is a column vector and $\mathbf{0}$ is a $1 \times m$ vector of zeros. The states $\{1, 2, \ldots, m\}$ shall be transient, while the state $m+1$ is absorbing.

Throughout this paper e denotes a column vector with all components equal to one for which the dimension is determined by the context. The $m \times m$ matrix T is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $\eta + Te = 0$. The distribution $F(\cdot)$ is given by

$$F(x) = 1 - \alpha \exp(Tx)e, \quad x \geqslant 0. \tag{2.1}$$

The following observations about PH distributions are important.

- $F(x)$ has a jump of height α_{m+1} at $x = 0$, and its density function is given by

$$f(x) = \alpha \exp(Tx)\eta, \quad x > 0. \tag{2.2}$$

- The Laplace–Stieltjes transform (LST) of $F(x)$ is given by

$$\widetilde{F}(s) = \alpha_{m+1} + \alpha(s \cdot I - T)^{-1}\eta \tag{2.3}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geqslant 0$, where I is the identity matrix of appropriate dimension.
- The kth noncentral moment of a PH distribution with representation (α, T) is given by

$$m_k = (-1)^k k! \alpha T^{-k} e, \quad k = 1, 2, \ldots. \tag{2.4}$$

Remark 2.2. There are some important special cases of PH distributions, for example, negative exponential distribution, Erlang distribution and Hyper-exponential distribution, etc. By setting $\alpha = (1)$ and $T = (-\lambda)$ in (2.1) gives the d.f. of the exponential distribution.

To formulate the following theorems, we need to define the Kronecker compositions of matrices.

Definition 2.3. If A and B are rectangular matrices of dimensions $m_1 \times m_2$ and $n_1 \times n_2$, respectively, their Kronecker product $A \otimes B$ is the matrix of dimensions $m_1 n_1 \times m_2 n_2$, written in compact form as $(a_{ij}B)$. The Kronecker sum of the square matrices C and D of orders p and q, respectively, is defined by $C \oplus D = C \otimes I_q + I_p \otimes D$, where I_k denotes the identity matrix of order k.

For more details about Kronecker compositions (see [10]), and for PH distributions (see [8]).

3. The kth moments of $X_{n:n}$ and $X_{1:n}$

Let $X_1, X_2, \ldots, X_n$ be independent nonidentically r.v.'s having PH d.f.'s:

$$F_i(x) = 1 - \alpha^{(i)} \exp(T^{(i)}x)e, \quad x \geqslant 0 \tag{3.1}$$

for $i = 1, 2, \ldots, n$.

Theorem 3.1. *For $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$*

$$\mu_{n:n}^{(k)} = \sum_{j=1}^{n} (-1)^{j+1} \mathbf{L}_j \tag{3.2}$$

and

$$\mu_{1:n}^{(k)} = \mathbf{L}_n, \tag{3.3}$$

where

$$\mathbf{L}_j = \sum \cdots \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n} (-1)^k k! \left(\alpha^{(i_1)} \otimes \alpha^{(i_2)} \otimes \cdots \otimes \alpha^{(i_j)}\right) \left(T^{(i_1)} \oplus T^{(i_2)} \oplus \cdots \oplus T^{(i_j)}\right)^{-k} e. \tag{3.4}$$

Proof. It is easily verified that the d.f. of $X_{n:n}$ is given by

$$\begin{aligned} F_{n:n}(x) &= \prod_{i=1}^{n} F_i(x) = \prod_{i=1}^{n}\left(1-\alpha^{(i)}\exp(T^{(i)}x)e\right) \\ &= 1-\sum_{i=1}^{n}\alpha^{(i)}\exp\left(T^{(i)}x\right)e+\sum_{1\leqslant i_1<i_2\leqslant n}\sum\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\right)\exp\left[\left(T^{(i_1)}\oplus T^{(i_2)}\right)x\right]e \\ &\quad -\sum_{1\leqslant i_1<i_2<i_3\leqslant n}\sum\sum\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\otimes\alpha^{(i_3)}\right)\exp\left[\left(T^{(i_1)}\oplus T^{(i_2)}\oplus T^{(i_3)}\right)x\right]e \\ &\quad +\cdots+(-1)^n\left(\alpha^{(1)}\otimes\alpha^{(2)}\otimes\cdots\otimes\alpha^{(n)}\right)\exp\left[\left(T^{(1)}\oplus T^{(2)}\oplus\cdots\oplus T^{(n)}\right)x\right]e, \end{aligned}$$

where we have used the relation $(A\exp(Bx))\otimes(C\exp(Dx)) = (A\otimes C)\exp[(B\oplus D)x]$, for matrices A, B, C and D of appropriate dimensions (see [10]).

Now on using $\int_0^\infty e^{-sx}\exp\left(T^{(i)}x\right)dx = \left(s\cdot I-T^{(i)}\right)^{-1}$ for all $s\neq 0$, the LST of $F_{n:n}(x)$ is given by

$$\begin{aligned} \widetilde{F}_{n:n}(s) &= \sum_{i=1}^{n}\left[1-\alpha^{(i)}e-\alpha^{(i)}\left(sI-T^{(i)}\right)^{-1}T^{(i)}e\right]-\sum_{1\leqslant i_1<i_2\leqslant n}\sum\left[\beta_{(2)}-\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\right)\left(sI-T_{(2)}\right)^{-1}T_{(2)}e\right] \\ &\quad +\sum_{1\leqslant i_1<i_2<i_3\leqslant n}\sum\sum\left[\beta_{(3)}-\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\otimes\alpha^{(i_3)}\right)\left(sI-T_{(3)}\right)^{-1}T_{(3)}e\right] \\ &\quad +\cdots+(-1)^{n+1}\left[\beta_{(n)}-\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\otimes\cdots\otimes\alpha^{(i_n)}\right)\left(sI-T_{(n)}\right)^{-1}T_{(n)}e\right], \end{aligned}$$

where $\beta_{(p)} = 1-\left(\alpha^{(i_1)}\otimes\alpha^{(i_2)}\otimes\cdots\otimes\alpha^{(i_p)}\right)e$ and $T_{(p)} = T^{(i_1)}\oplus T^{(i_2)}\oplus\cdots\oplus T^{(i_p)}$, $p = 2, 3, \ldots, n$.

Differentiating $\widetilde{F}_{n:n}(s)$ k times with respect to s and letting $s = 0$, yields

$$\mu_{n:n}^{(k)} = E\left(X_{n:n}^{(k)}\right) = (-1)^k\frac{d^k}{ds^k}\widetilde{F}_{n:n}(s)\bigg|_{s=0}.$$

Hence, by using

$$\frac{d^k}{ds^k}(s\cdot I-T)^{-1} = (-1)^k k!\,(s\cdot I-T)^{-(k+1)},$$

we obtain (3.2).

Using the fact

$$\begin{aligned} F_{1:n}(x) &= 1-\prod_{i=1}^{n}(1-F_i(x)) = 1-\prod_{i=1}^{n}\left(\alpha^{(i)}\exp(T^{(i)}x)e\right) \\ &= 1-\left(\alpha^{(1)}\otimes\alpha^{(2)}\otimes\cdots\otimes\alpha^{(n)}\right)\exp\left[\left(T^{(1)}\oplus T^{(2)}\oplus\cdots\oplus T^{(n)}\right)x\right]e, \end{aligned}$$

by the same method as in the proof of (3.2), we get (3.3) and the proof is completed. □

Remark 3.2. Theorem 3.1 shows that $\mu_{n:n}^{(k)}$ and $\mu_{1:n}^{(k)}$ can be expressed simply in terms of the inverse and exponential of matrices. These calculations can be computed relatively easily using commercial software such as MATLAB.

Remark 3.3. For the case when the X_i's are independent and nonidentical r.v.'s having exponential distribution (that is, $F_i(x) = 1-e^{-\lambda_i x}$, $x>0$, $i = 1, 2, \ldots, n$), the $\mathbf{L}_j$ in Theorem 3.1 reduces to

$$\mathbf{L}_j = \sum_{1\leqslant i_1<i_2<\cdots<i_j\leqslant n}\cdots\sum\frac{k!}{(\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_j})^k},\quad j = 1, 2, \ldots, n. \tag{3.5}$$

4. Recurrence relation

In what follows, we need to introduce some additional notations. Denote by Per(A) the permanent of an n-by-n matrix $A = (a_{i,j})$, defined as

$$\mathrm{Per}(A) = \sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i,\sigma(i)},$$

where S_n denotes the set of permutations of $1, 2, \ldots, n$.

The permanent remains unchanged if the rows or columns of the matrix are permuted. Furthermore the permanent admits a Laplace expansion along any row or column of the matrix. Thus if we denote by $A(i, j)$ the matrix obtained by deleting row i and column j of the n-by-n matrix A, then

$$\mathrm{Per}(A) = \sum_{j=1}^{n} a_{ij}\mathrm{Per}(A(i, j)), \quad i = 1, 2, \ldots, n$$

and

$$\mathrm{Per}(A) = \sum_{i=1}^{n} a_{ij}\mathrm{Per}(A(i, j)), \quad j = 1, 2, \ldots, n.$$

One can refer to [11] for a detailed survey on permutations.

If $a_1, a_1, \ldots$ are column vectors, then

$$\begin{array}{ccc} [a_1 & a_2 & \cdots] \\ i_1 & i_2 & \cdots \end{array}$$

will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on.

It is well known that the d.f. of the rth o.s. $X_{r:n} (1 \leqslant r \leqslant n)$ can be written as (see [6])

$$F_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!(n-i)!} \mathrm{Per} \begin{array}{c} \begin{bmatrix} F_1(x) & 1 - F_1(x) \\ \vdots & \vdots \\ F_n(x) & 1 - F_n(x) \end{bmatrix} \\ \begin{array}{cc} i \qquad & \qquad n - i \end{array} \end{array}. \tag{4.1}$$

In the following, we derive a recurrence relation satisfied by the moments of o.s. from the PH distribution by making use of the equation in (4.1).

Theorem 4.1. *For $r = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$*

$$\mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + \sum_{j=1}^{r} (-1)^{j-1} \binom{n-r+j}{j-1} \mathbf{L}_{n-r+j}, \tag{4.2}$$

where the sequence $\{\mathbf{L}_j\}_{j=1}^{j=r}$ is given by (3.4) and $\mu_{0:n}^{(k)} = 0$ for convention.

Proof. From (4.1), we have that, for any $1 < r \leqslant n$,

$$F_{r-1:n}(x) = F_{r:n}(x) + \frac{1}{(r-1)!(n-r+1)!} \mathrm{Per} \begin{array}{c} \begin{bmatrix} F_1(x) & 1 - F_1(x) \\ \vdots & \vdots \\ F_n(x) & 1 - F_n(x) \end{bmatrix} \\ \begin{array}{cc} r - 1 & n - r + 1 \end{array} \end{array} \tag{4.3}$$

which can be expressed as

$$F_{r-1:n}(x) = F_{r:n}(x) + \sum_{\mathscr{P}} \prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=1}^{n-r+1} \left(1 - F_{i_{n-j+1}}(x)\right), \tag{4.4}$$

where the summation $\mathscr{P}$ extends over all permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ for which $1 \leqslant i_1 < i_2 < \cdots < i_{r-1} \leqslant n$ and $1 \leqslant i_r < i_{r+1} < \cdots < i_n \leqslant n$.

By (3.1), we have

$$\begin{aligned} \prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=1}^{n-r+1} \left(1 - F_{i_{n-j+1}}(x)\right) &= \prod_{j=1}^{r-1} \left(1 - \alpha^{(i_j)} \exp(T^{(i_j)}x)e\right) \prod_{j=r}^{n} \alpha^{(i_j)} \exp(T^{(i_j)}x)e \\ &= \Bigg\{ 1 - \sum_{j_1=1}^{r-1} \alpha^{(i_{j_1})} \exp\left(T^{(i_{j_1})}x\right) e + \sum\sum_{1\leqslant j_1<j_2\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})}\right) \exp\left[\left(T^{(i_{j_1})} \oplus T^{(i_{j_2})}\right) x\right] e \\ &\quad - \sum\sum\sum_{1\leqslant j_1<j_2<j_3\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha^{(i_{j_3})}\right) \exp\left[\left(T^{(i_{j_1})} \oplus T^{(i_{j_2})} \oplus T^{(i_{j_3})}\right) x\right] e \\ &\quad + \cdots + (-1)^{r-1} \left(\alpha^{(i_1)} \otimes \alpha^{(i_2)} \otimes \cdots \otimes \alpha^{(i_{r-1})}\right) \exp\left[\left(T^{(i_1)} \oplus T^{(i_2)} \oplus \cdots \oplus T^{(i_{r-1})}\right) x\right] e \Bigg\} \prod_{j=r}^{n} \alpha^{(i_j)} \exp(T^{(i_j)}x)e \end{aligned}$$

$$= \alpha_r \exp(T_r x)\, e - \sum_{j_1=1}^{r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha_r\right) \exp\left[\left(T^{(i_{j_1})} \oplus T_r\right) x\right] e$$
$$+ \sum\sum_{1\leqslant j_1<j_2\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha_r\right) \exp\left[\left(T^{(i_{j_1})} \oplus T^{(i_{j_2})} \oplus T_r\right) x\right] e$$
$$- \sum\sum\sum_{1\leqslant j_1<j_2<j_3\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha^{(i_{j_3})} \otimes \alpha_r\right) \exp\left[\left(T^{(i_{j_1})} \oplus T^{(i_{j_2})} \oplus T^{(i_{j_3})} \oplus T_r\right) x\right] e$$
$$+ \cdots + (-1)^{r-1} \left(\alpha^{(i_1)} \otimes \alpha^{(i_2)} \otimes \cdots \otimes \alpha^{(i_n)}\right) \exp\left[\left(T^{(i_1)} \oplus T^{(i_2)} \oplus \cdots \oplus T^{(i_n)}\right) x\right] e, \tag{4.5}$$

where $\alpha_r = \alpha^{(i_r)} \otimes \alpha^{(i_{r+1})} \otimes \cdots \otimes \alpha^{(i_n)}$, and $T_r = T^{(i_r)} \oplus T^{(i_{r+1})} \oplus \cdots \oplus T^{(i_n)}$.

For convenience, denote

$$G_r(x) = \sum_{\mathcal{P}} \prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=1}^{n-r+1} \left(1 - F_{i_{n-j+1}}(x)\right). \tag{4.6}$$

Denote by $\widetilde{F}_{r-1:n}(s)$ and $\widetilde{F}_{r:n}(s)$, respectively, the LST's of $F_{r-1:n}(x)$ and $F_{r:n}(x)$. It is easy to see from (4.4) that

$$\widetilde{F}_{r-1:n}(s) = \int_0^\infty \mathrm{e}^{-sx} \mathrm{d}F_{r-1:n}(x) = \widetilde{F}_{r:n}(s) + \phi_r(s), \tag{4.7}$$

where

$$\phi_r(s) = \int_0^\infty \mathrm{e}^{-sx} \mathrm{d}G_r(x). \tag{4.8}$$

Substituting (4.5) and (4.6) into (4.8), together with (2.3), we then yield that

$$\phi_r(s) = -\sum_{\mathcal{P}} \Bigg(\left[1 - \alpha_r e - \alpha_r (sI - T_r)^{-1} T_r e\right] - \sum_{j_1=1}^{r-1} \left[\gamma_{(1)} - \left(\alpha^{(i_{j_1})} \otimes \alpha_r\right) \left(sI - S_{(1)}\right)^{-1} S_{(1)} e\right]$$
$$+ \sum\sum_{1\leqslant j_1<j_2\leqslant r-1} \left[\gamma_{(2)} - \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha_r\right) \left(sI - S_{(2)}\right)^{-1} S_{(2)} e\right]$$
$$- \sum\sum\sum_{1\leqslant j_1<j_2<j_3\leqslant r-1} \left[\gamma_{(3)} - \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha^{(i_{j_3})} \otimes \alpha_r\right) \left(sI - S_{(3)}\right)^{-1} S_{(3)} e\right]$$
$$+ \cdots + (-1)^{r-1} \left[\gamma_{(r-1)} - \left(\alpha^{(i_1)} \otimes \alpha^{(i_2)} \otimes \cdots \otimes \alpha^{(i_n)}\right) \left(sI - S_{(r-1)}\right)^{-1} S_{(r-1)} e\right] \Bigg), \tag{4.9}$$

where $\gamma_{(q)} = 1 - \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \cdots \otimes \alpha^{(i_{j_q})} \otimes \alpha_r\right) e$ and $S_{(q)} = T^{(i_{j_1})} \oplus T^{(i_{j_2})} \oplus \cdots \oplus T^{(i_{j_q})} \oplus T_r,\ q = 1, 2, \ldots, r-1$.

Differentiating (4.7) k times with respect to s and letting $s = 0$, yields

$$\mu_{r-1:n}^{(k)} = \mu_{r:n}^{(k)} + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \phi_r(s)\Big|_{s=0}. \tag{4.10}$$

Furthermore, it follows easily from (4.9) that

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \phi_r(s)\Big|_{s=0} = (-1)^{k+1} k! \sum_{\mathcal{P}} \Bigg\{ \alpha^{(r)} \left(T^{(r)}\right)^{-k} e - \sum_{j_1=1}^{r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha_r\right) \left(S_{(1)}\right)^{-k} e$$
$$+ \sum\sum_{1\leqslant j_1<j_2\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha_r\right) \left(S_{(2)}\right)^{-k} e$$
$$- \sum\sum\sum_{1\leqslant j_1<j_2<j_3\leqslant r-1} \left(\alpha^{(i_{j_1})} \otimes \alpha^{(i_{j_2})} \otimes \alpha^{(i_{j_3})} \otimes \alpha_r\right) \left(S_{(3)}\right)^{-k} e$$
$$+ \cdots + (-1)^{r-1} \left(\alpha^{(i_1)} \otimes \alpha^{(i_2)} \otimes \cdots \otimes \alpha^{(i_n)}\right) \left(S_{(r-1)}\right)^{-k} e \Bigg\}$$

Table 1
The single moments of o.s. from the PH distribution.

$\mu_{n:n}^{(k)} =$	$\mathbf{L}_1$	$-\mathbf{L}_2+$	$\cdots$	$(-1)^{n-3}\mathbf{L}_{n-4}$	$(-1)^{n-2}\mathbf{L}_{n-3}$	$(-1)^{n-1}\mathbf{L}_{n-2}$	$(-1)^{n}\mathbf{L}_{n-1}$	$(-1)^{n+1}\mathbf{L}_{n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\mu_{5:n}^{(k)} =$				$\mathbf{L}_{n-4}$	$(4-n)\mathbf{L}_{n-3}$	$\frac{(n-3)(n-4)}{2!}\mathbf{L}_{n-2}$	$-\frac{(n-2)(n-3)(n-4)}{3!}\mathbf{L}_{n-1}$	$\frac{(n-1)(n-2)(n-3)(n-4)}{4!}\mathbf{L}_{n}$
$\mu_{4:n}^{(k)} =$					$\mathbf{L}_{n-3}$	$(3-n)\mathbf{L}_{n-2}$	$\frac{(n-2)(n-3)}{2!}\mathbf{L}_{n-1}$	$-\frac{(n-1)(n-2)(n-3)}{3!}\mathbf{L}_{n}$
$\mu_{3:n}^{(k)} =$						$\mathbf{L}_{n-2}$	$(2-n)\mathbf{L}_{n-1}$	$\frac{(n-1)(n-2)}{2!}\mathbf{L}_{n}$
$\mu_{2:n}^{(k)} =$							$\mathbf{L}_{n-1}$	$(1-n)\mathbf{L}_{n}$
$\mu_{1:n}^{(k)} =$								$\mathbf{L}_{n}$

which further implies that

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k}\phi_r(s)\bigg|_{s=0} = -\sum_{j=1}^{r}(-1)^{j-1}C_j(r,n)\mathbf{L}_{n-r+j},$$

where $\{\mathbf{L}_j\}_{j=n-r+1}^{j=n}$ is given by (3.4), and $C_j(r,n)$ is a suitable sequence of constants.

Now, by virtue of the relations $\sum_{\mathcal{P}}(1) = \binom{n}{r-1}$ and $\sum\cdots\sum_{1\leqslant i_1<i_2<\cdots<i_m\leqslant n}(1) = \binom{n}{m}$, for all $m \leqslant n$, we have

$$\binom{n}{r-1}\binom{r-1}{j-1} = C_j(r,n)\binom{n}{n-r+j}.$$

Hence,

$$C_j(r,n) = \binom{n-r+j}{j-1}$$

which completes the proof. $\square$

Remark 4.2. Theorem 4.1 will enable one to compute all the single moments of all o.s. in a simple recursive way. Specifically, $\mu_{r:n}^{(k)}(1 \leqslant r \leqslant n)$ could be derived recursively by using (4.2), starting with $\mu_{1:n}^{(k)}$. Table 1 presents all the kth moments of all o.s.

5. Applications

PH distributions have been studied extensively in the past several decades, due to their vast applications in a variety of fields. There are also many practical applications for the moments of order statistics from PH distributions. These moments have wide applications in stochastic activity networks (SANs). A SAN is a directed acyclic network $G = (V, A)$, where $V = \{1, 2, \ldots, n\}$ is the set of nodes representing the project events and A is the set of arcs representing the project activities. Let the SAN possess a unique source node 1 and a unique sink node n. With each network activity $(i, j) \in A$ denoted as a_{ij}, we associate its duration X_{ij}, which is a non-negative random variable, and further we assume that all X_{ij}'s are mutually independent.

It is well known that one of the most important problems in the analysis of SANs is to find the d.f.'s of the following two r.v.'s:

$$X_{n:n} = \max(X_1, X_2, \ldots, X_n) \quad \text{and} \quad X_{1:n} = \min(X_1, X_2, \ldots, X_n)$$

i.e., finding $P(X_{n:n} \leqslant t)$ and $P(X_{1:n} \leqslant t)$ or finding the kth moments of $X_{n:n}$ and $X_{1:n}$, where the r.v.'s $X_1, X_2, \ldots, X_n$ represent the activity durations in SANs.

It is easy to show that the project completion time (time of realization of node n) is given by the length of a longest $(1, n)$ path in G. The moments of order statistics from PH distributions can be used to estimate the project completion time when each activity represented by PH r.v.'s. The moments method for estimating the project completion time of the above networks is particularly useful when activity time distributions are in empirical form or when the analytical method cannot be used because the mathematics becomes unmanageable.

Denote by v_j the mean completion time of the node j. Then the successive v_j can now be derived recursively as follows:

$$\begin{aligned} &v_1 = 0, \\ &v_j = E\max(v_i + X_{ij};\, i \in B_j), \quad j = 2, \ldots, n \end{aligned} \tag{5.1}$$

where $B_j = \{i \in V, (i, j) \in A\}$ and X_{ij} denotes the duration of activity (i, j) which is assumed to be PH distributed. The other moments can be easily derived by using (3.2) and (5.1).

Acknowledgements

The authors would like to thank the anonymous referees and editors for their helpful suggestions that lead to significant improvement of this paper.

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